Quantum measurement bounds beyond the uncertainty relations

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We give a bound to the precision in the estimation of a parameter in terms of the expectation value of an observable. It is an extension of the Cramér-Rao inequality and of the Heisenberg uncertainty relation, where the estimation precision is typically bounded in terms of the variance of an observable.

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Quantum measurements are limited by bounds such as the Heisenberg uncertainty relations [1, 2] or the quantum Cramér-Rao inequality [3–6], which typically constrain the ability in recovering a target quantity (e.g. a relative phase) through the *standard deviation* of a conjugate one (e.g. the energy) evaluated on the state of the probing system. Here we give a new bound related to the *expectation value*: we show that the precision in the quantity cannot scale better than the inverse of the expectation value (above a "ground state") of its conjugate counterpart. It is especially relevant in the expanding field of quantum metrology [7]: it settles in the positive the longstanding conjecture of quantum optics [8–13], recently challenged [14–16], that the ultimate phase-precision limit in interferometry is lower bounded by the inverse of the total number of photons employed in the estimation process.

The aim of Quantum Parameter Estimation [3–6] is to recover the *unknown* value x of a parameter that is written into the state ρ_x of a probe system through some *known* encoding mechanism U_x. For example, we can recover the relative optical delay x among the two arms of a Mach-Zehnder interferometer described by its unitary evolution U_x using as probe a light beam fed into the interferometer. The statistical nature of quantum mechanics induces fluctuations that limit the ultimate precision which can be achieved (although we can exploit quantum "tricks" such as entanglement and squeezing in optimizing the state preparation of the probe and/or the detection stage [17]). In particular, if the encoding stage is repeated several times using ν identical copies of the same probe input state ρ_x, the root mean square error (RMSE) ΔX of the resulting estimation process is limited by the quantum Cramér-Rao bound [3–6] $\Delta X \geqslant 1/\sqrt{\nu Q(x)}$, where $Q(x)$ is the quantum Fisher information. For pure probe states and unitary encoding mechanism U_x, $Q(x)$ is equal to the variance $(\Delta H)^2$ (calculated on the probe state) of the generator H of the transformation $U_x = e^{-ixH}$. In this case, the Cramér-Rao bound takes the form

$$\Delta X \geqslant 1/(\sqrt{\nu}\Delta H) \tag{1}$$

of an uncertainty relation [5, 6]. In fact, if the parameter x can be connected to an observable, Eq. (1) corresponds to the Heisenberg uncertainty relation for conjugate variables [1, 2]. This bound is asymptotically achievable in the limit of $\nu \to \infty$ [3, 4].

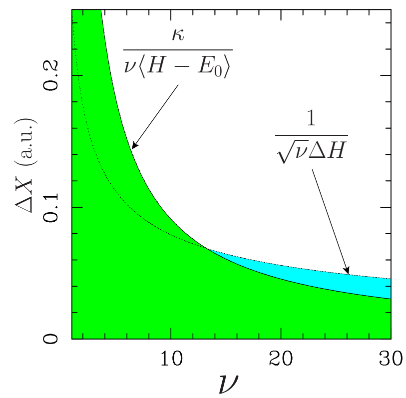

FIG. 1: Lower bounds to the precision estimation ΔX as a function of the experimental repetitions ν. The green area in the graph represents the forbidden values due to our bound (2). The blue (dashed-line) area represents the forbidden values due to the Cramér-Rao bound, or the Heisenberg uncertainty, (1). Possible estimation strategies have precision ΔX that cannot penetrate in the colored regions. For large ν the Cramér-Rao bound (which scales as $1/\sqrt{\nu}$) is stronger, as expected since in this regime it is achievable. Our bound is not achievable in general, so that the green area may be expanded when considering specific estimation strategies. [Here we used $\langle H\rangle - E_0 = 0.1$ (a.u.) and $\Delta H = 4$ (a.u.).]

Here we will derive a bound in terms of the expectation value of H, which (in the simple case of constant ΔX) takes the form (see Fig. 1)

$$\Delta X \geqslant \kappa/[\nu(\langle H\rangle - E_0)] \;, \tag{2}$$

where E_0 is the value of a "ground state", the minimum eigenvalue of H whose eigenvector is populated in the probe state (e.g. the ground state energy when H is the probe's Hamiltonian), and $\kappa \simeq 0.091$ is a constant of order one. Our bound holds both for biased and unbiased measurement procedures, and for pure and mixed probe states. When ΔX is dependent on x, a constraint of the form (2) can be placed on the average value of $\Delta X(x)$ evaluated on any two values x and x' of the parameter

which are sufficiently separated, namely

$$\frac{\Delta X(x)+\Delta X(x')}{2} \;\geqslant\; \frac{\kappa}{\nu(\langle H\rangle - E_0)}\,. \tag{3}$$

Hence, we cannot exclude that strategies whose error ΔX depend on x may have a "sweet spot" where the bound (2) may be beaten [14], but inequality (3) shows that the average value of ΔX is subject to the bound. Thus, these strategies are of no practical use, since the sweet spot depends on the unknown parameter x to be estimated and the extremely good precision in the sweet spot must be counterbalanced by a correspondingly bad precision nearby.

Proving the bound (2) in full generality is clearly not a trivial task since no definite relation can be established between $\nu(\langle H\rangle - E_0)$ and the term $\sqrt{\nu}\Delta H$ on which the Cramér-Rao bound is based. In particular, scaling arguments on ν cannot be used since, on one hand, the value of ν for which Eq. (1) saturates is not known (except in the case in which the estimation strategy is fixed [8], which has little fundamental relevance) and, on the other hand, input probe states ρ whose expectation values $\langle H\rangle$ depend explicitly on ν may be employed, e.g. see Ref. [14]. To circumvent these problems our proof is based on the quantum speed limit [18], a generalization of the Margolus-Levitin [19] and Bhattacharyya bounds [20, 21] which links the fidelity F between the two joint states $\rho_x^{\otimes\nu}$ and $\rho_{x'}^{\otimes\nu}$ to the difference $x'-x$ of the parameters x and x' imprinted on the states through the mapping $U_x = e^{-ixH}$ [The fidelity between two states ρ and σ is defined as $F=\{\mathrm{Tr}[\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}]\}^2$. A connection between quantum metrology and the Margolus-Levitin theorem was proposed in [22], but this claim was subsequently retracted in [23].] In the case of interest here, the quantum speed limit [18] implies

$$|x'-x| \geqslant \frac{\pi}{2}\max\left[\frac{\alpha(F)}{\nu(\langle H\rangle - E_0)}\,,\;\frac{\beta(F)}{\sqrt{\nu}\Delta H}\right]\quad, \tag{4}$$

where the ν and $\sqrt{\nu}$ factors at the denominators arise from the fact that here we are considering ν copies of the probe states ρ_x and $\rho_{x'}$, and where $\alpha(F)\simeq\beta^2(F) = 4\arccos^2(\sqrt{F})/\pi^2$ are the functions plotted in Fig. 2 of the supplementary material. The inequality (4) tells us that the parameter difference $|x'-x|$ induced by a transformation $e^{-i(x'-x)H}$ which employs resources $\langle H\rangle - E_0$ and ΔH cannot be arbitrarily small (when the parameter x coincides with the evolution time, this sets a limit to the "speed" of the evolution, the quantum speed limit).

We now give the main ideas of the proof of (2) by focusing on a simplified scenario, assuming pure probe states $|\psi_x\rangle = U_x|\psi\rangle$, and unbiased estimation strategies constructed in terms of projective measurements with RSME ΔX that do not depend on x (all these assumptions are dropped in the supplementary material). For unbiased estimation, $x=\sum_j P_j(x)x_j$ and the RMSE coincides with the variance of the distribution $P_j(x)$, i.e. $\Delta X = \sqrt{\sum_j P_j(x)[x_j-x]^2}$, where $P_j(x) = |\langle x_j|\psi_x\rangle^{\otimes\nu}|^2$ is the probability of obtaining the result x_j while measuring the joint state $|\psi_x\rangle^{\otimes\nu}$ with a projective measurement on the joint basis $|x_j\rangle$. Let us consider two values x and x' of the parameter that are further apart than the measurement's RMSE, i.e. $x'-x = 2\lambda\Delta X$ with $\lambda > 1$. If no such x and x' exist, the estimation is extremely poor: basically the whole domain of the parameter is smaller than the RMSE. Hence, for estimation strategies that are sufficiently accurate to be of interest, we can always assume that such a choice is possible (see below). The Tchebychev inequality states that for an arbitrary probability distribution p, the probability that a result x lies more than $\lambda\Delta X$ away from the average μ is upper bounded by $1/\lambda^2$, namely $p(|x-\mu|\geqslant\lambda\Delta X)\leqslant 1/\lambda^2$. It implies that the probability that measuring $|\Psi_{x'}\rangle := |\psi_{x'}\rangle^{\otimes\nu}$ the outcome x_j lies within $\lambda\Delta X$ of the mean value associated with $|\Psi_x\rangle := |\psi_x\rangle^{\otimes\nu}$ cannot be larger $1/\lambda^2$. By the same reasoning, the probability that measuring $|\Psi_x\rangle$ the outcome x_j will lie within $\lambda\Delta X$ of the mean value associated with $|\Psi_{x'}\rangle$ cannot be larger $1/\lambda^2$. This implies that the overlap between the states $|\Psi_x\rangle$ and $|\Psi_{x'}\rangle$ cannot be too large: more precisely, $F=|\langle\Psi_x|\Psi_{x'}\rangle|^2\leqslant 4/\lambda^2$. Replacing this expression into (4) (exploiting the fact that α and β are decreasing functions) we obtain

$$2\lambda\Delta X \geqslant \frac{\pi}{2}\max\left[\frac{\alpha(4/\lambda^2)}{\nu(\langle H\rangle - E_0)}\,,\;\frac{\beta(4/\lambda^2)}{\sqrt{\nu}\Delta H}\right]\;, \tag{5}$$

whence we obtain (2) by optimizing over λ the first term of the max, i.e. choosing $\kappa = \sup_\lambda \pi\,\alpha(4/\lambda^2)/(4\lambda)\simeq 0.091$. The second term of the max gives rise to a quantum Cramér-Rao type uncertainty relation (or a Heisenberg uncertainty relation) which, consistently with the optimality of Eq. (1) for $\nu\gg 1$, has a pre-factor $\pi\beta(4/\lambda^2)/(4\lambda)$ which is smaller than 1 for all λ. This means that for large ν the bound (2) will be asymptotically superseded by the Cramér-Rao part, which scales as $\propto 1/\sqrt{\nu}$ and is achievable in this regime.

Analogous results can be obtained (see supplementary material) when considering more general scenarios where the input states of the probes are not pure, the estimation process is biased, and it is performed with arbitrary POVM measurements. (In the case of biased measurements, the constant κ in (2) and (3) must be replaced by $\kappa=\sup_\lambda \pi\alpha(4/\lambda^2)/[4(\lambda+1)]\simeq 0.074$, where a $+1$ term appears in the denominator.) In this generalized context, whenever the RMSE depends explicitly on the value x of the parameter, the result (2) derived above is replaced by the weaker relation (3). Such inequality clearly does not necessarily exclude the possibility that at a "sweet spot" the estimation might violate the scaling (2). However, Eq. (3) is still sufficient strong to exclude accuracies of the form $\Delta X(x) = 1/R(x,\nu\langle H\rangle)$ where, as in Refs. [14, 24], $R(x,z)$ is a function of z which, for all x,

increases more than linearly, i.e. $\lim_{z\to\infty} z/R(x,z) = 0$.

The bound (2) has been derived under the explicit assumption that x and x' exists such that $x' - x \geqslant 2\lambda\Delta X$ for some $\lambda > 1$, which requires one to have $x' - x \geqslant 2\Delta X$. This means that the estimation strategy must be good enough: the probe is sufficiently sensitive to the transformation U_x that it is shifted by more than ΔX during the interaction. The existence of pathological estimation strategies which violate such condition cannot be excluded *a priori*. Indeed trivial examples of this sort can be easily constructed, a fact which may explain the complicated history of the Heisenberg bound with claims [8–13] and counterclaims [14–16, 24]. It should be stressed however, that the assumption $x' - x \geqslant 2\Delta X$ is always satisfied except for extremely poor estimation strategies with such large errors as to be practically useless. One may think of repeating such a poor estimation strategy $\nu > 1$ times and of performing a statistical average to decrease its error. However, for sufficiently large ν the error will decrease to the point in which the ν repetitions of the poor strategy are, collectively, a good strategy, and hence again subject to our bounds (2) and (3).

Our findings are particularly relevant in the field of quantum optics, where a controversial and longly debated problem [8–16, 24] is to determine the scaling of the ultimate limit in the interferometric precision of estimating a phase as a function of the energy $\langle H\rangle$ devoted to preparing the ν copies of the probes: it has been conjectured [8–13] that the phase RMSE is lower bounded by the inverse of the total number of photons employed in the experiment, the "Heisenberg bound" for interferometry[1]. Its achievability has been recently proved [25], and, in the context of quantum parameter estimation, it corresponds to an equation of the form of Eq. (2), choosing $x = \phi$ (the relative phase between the modes in the interferometer) and $H = a^\dagger a$ (the number operator). The validity of this bound has been questioned several times [14–16, 24]. In particular schemes have been proposed [14, 24] that apparently permit better scalings in the achievable RMSE (for instance $\Delta X \approx (\nu\langle H\rangle)^{-\gamma}$ with $\gamma > 1$). None of these protocols have conclusively proved such scalings for arbitrary values of the parameter x, but a sound, clear argument against the possibility of breaking the $\gamma = 1$ scaling of Eq. (2) was missing up to now. Our results validate the Heisenberg bound by showing that it applies to all those estimation strategies whose RMSE ΔX do not depend on the value of the parameter x, and that the remaining strategies can only have good precision for isolated (hence practically useless) values of the unknown parameter x.

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Supplementary material

Our bound refers to the estimation of the value x of a real parameter X that identifies a unitary transformation $U_x = e^{-iHx}$, generated by an Hermitian operator H. The usual setting in quantum channel parameter estimation (see [7] for a recent review) is to prepare ν copies of a probe system in a fiducial state ρ, apply the mapping U_x to each of them as $\rho \to \rho_x = U_x \rho U_x{}^\dagger$, and then perform a (possibly joint) measurement on the joint output state $\rho_x^{\otimes\nu}$, the measurement being described by a generic Positive Operator-Valued Measure (POVM) of elements $\{E_j\}$. [The possibility of applying a joint transformation on the ν probes before the interaction U_x (e.g. to entangle them as studied in [17]) can also be considered, but it is useless in this context, since it will not increase the linear scaling in ν of the term $\nu(\langle H\rangle - E_0)$ that governs our bounds.] The result j of the measurement is finally used to recover the quantity x through some data processing which assigns to each outcome j of the POVM a value x_j which represents the estimation of x. The accuracy of the process can be gauged by the RMSE of the problem, i.e. by the quantity

$$\Delta X := \sqrt{\sum_j P_j(x)[x_j - x]^2} = \sqrt{\delta^2 X + (\bar{x} - x)^2}, \quad (6)$$

where $P_j(x) = \mathrm{Tr}[E_j \rho_x^{\otimes\nu}]$ is the probability of getting the outcome j when measuring $\rho_x^{\otimes\nu}$, $\bar{x} := \sum_j P_j(x) x_j$ is the average of the estimator function, and where

$$\delta^2 X := \sum_j P_j(x)[x_j - \bar{x}]^2 \ , \quad (7)$$

is the variance of the random variable x_j. The estimation is said to be unbiased if $\bar{x}$ coincides with the real value x, i.e. $\bar{x} = x$, so that, in this case, ΔX coincides with δX. General estimators however may be biased with $\bar{x} \neq x$, so that $\Delta X > \delta X$ (in this case, they are called asymptotically unbiased if $\bar{x}$ converges to x in the limit $\nu \to \infty$).

In the main text we restricted our analysis to pure states of the probe $\rho = |\psi\rangle\langle\psi|$ and focused on projective measurements associated to unbiased estimation procedures whose RMSE ΔX is independent on x. Here we

[1] This "Heisenberg" bound [8–13] should not be confused with the Heisenberg scaling defined for general quantum estimation problem [7] in which the $\sqrt{\nu}$ at the denominator of Eq. (1) is replaced by ν by feeding the ν inputs with entangled input states – e.g. see Ref. [7, 17].

extend the proof to drop the above simplifying assumptions, considering a generic (non necessarily unbiased) estimation process which allows one to determine the value of the real parameter X associated with the non necessarily pure input state ρ.

Take two values x and x' of X such that their associated RMSE verifies the following constraints

$$\Delta X(x) \neq 0 \;, \tag{8}$$
$$|x - x'| = (\lambda+1)[\Delta X(x) + \Delta X(x')] \;, \tag{9}$$

for some fixed value λ greater than 1 (the right hand side of Eq. (9) can be replaced by $\lambda[\Delta X(x)+\Delta X(x')]$ if the estimation is unbiased). In these expressions $\Delta X(x)$ and $\Delta X(x')$ are the RMSE of the estimation evaluated through Eq. (6) on the output states $\rho_x^{\otimes\nu}$ and $\rho_{x'}^{\otimes\nu}$ respectively (to include the most general scenario we do allow them to depend explicitly on the values taken by the parameter X). In the case in which the estimation is asymptotically unbiased and the quantum Fisher information $Q(x)$ of the problem takes finite values, the condition (8) is always guaranteed by the quantum Cramér-Rao bound [3–6] (but notice that our proof holds also if the quantum Cramér-Rao bound does not apply – in particular, we do not require the estimation to be asymptotically unbiased). The condition (9) on the other hand is verified by any estimation procedure which achieves a reasonable level of accuracy: indeed, if it is not verified, then this implies that the interval over which X can span is not larger than twice the average RMSE achievable in the estimation.

Since the fidelity between two quantum states is the minimum of the classical fidelity of the probability distributions from arbitrary POVMs [26], we can bound the fidelity between $\rho_x^{\otimes\nu}$ and $\rho_{x'}^{\otimes\nu}$ as follows

$$F := \Big[\mathrm{Tr}\sqrt{\sqrt{\rho_x^{\otimes\nu}}\,\rho_{x'}^{\otimes\nu}\sqrt{\rho_x^{\otimes\nu}}}\Big]^2 \leqslant \Big[\sum_j \sqrt{P_j(x)P_j(x')}\Big]^2 \;, \tag{10}$$

with $P_j(x) = \mathrm{Tr}[E_j\rho_x^{\otimes\nu}]$ and $P_j(x') = \mathrm{Tr}[E_j\rho_{x'}^{\otimes\nu}]$. The right-hand-side of this expression can be bound as

$$\begin{aligned}\sum_j \sqrt{P_j(x)P_j(x')} &= \sum_{j\in I}\sqrt{P_j(x)P_j(x')} + \sum_{j\notin I}\sqrt{P_j(x)P_j(x')} \\ &\leqslant \sqrt{\sum_{j\in I}P_j(x)\sum_{j'\in I}P_{j'}(x')} + \sqrt{\sum_{j\notin I}P_j(x)\sum_{j'\notin I}P_{j'}(x')} \\ &\leqslant \sqrt{\sum_{j\in I}P_j(x)} + \sqrt{\sum_{j'\notin I}P_{j'}(x')} \;,\end{aligned} \tag{11}$$

where I is a subset of the domain of possible outcomes j that we will specify later, and where we used the Cauchy-Schwarz inequality and the fact that $\sum_{j'\in I}P_{j'}(x') \leqslant 1$ and $\sum_{j\notin I}P_j(x) \leqslant 1$ independently from I. Now, take I to be the domain of the outcomes j such that

$$|x_j - \bar{x}'| \leqslant \lambda\delta X', \tag{12}$$

where λ is a positive parameter (here $\bar{x}'$ and $(\delta X')^2$ are the average and the variance value of x_j computed with the probability distribution $P_j(x')$). From the Tchebychev inequality it then follows that

$$\sum_{j'\notin I} P_{j'}(x') \leqslant 1/\lambda^2 \;, \tag{13}$$

which gives a significant bound only when $\lambda > 1$. To bound the other term on the rhs of Eq. (11) we notice that $|x-x'| \leqslant \big|x-\bar{x}\big| + \big|x'-\bar{x}'\big| + \big|\bar{x}-\bar{x}'\big|$ and use Eq. (9) and (6) to write

$$\begin{aligned}\big|\bar{x}-\bar{x}'\big| &\geqslant (\lambda+1)(\Delta X + \Delta X') - \big|x-\bar{x}\big| - \big|x'-\bar{x}'\big| \\ &= (\lambda+1)(\Delta X + \Delta X') - \sqrt{\Delta^2 X - \delta^2 X} - \sqrt{\Delta^2 X' - \delta^2 X'} \geqslant \lambda(\Delta X + \Delta X') \;.\end{aligned} \tag{14}$$

From Eq. (12) we also notice that for $j \in I$ we have

$$\big|\bar{x}-\bar{x}'\big| \leqslant \big|\bar{x}-x_j\big| + \big|x_j - \bar{x}'\big| \leqslant \big|\bar{x}-x_j\big| + \lambda\delta X' \;, \tag{15}$$

which with the previous expression gives us

$$\big|\bar{x}-x_j\big| \geqslant \lambda(\Delta X + \Delta X') - \lambda\delta X' \geqslant \lambda\Delta X \geqslant \lambda\delta X \;, \tag{16}$$

and hence (using again the Tchebychev inequality)

$$\sum_{j\in I} P_j(x) \leqslant 1/\lambda^2 \ . \tag{17}$$

Replacing (13) and (17) into (10) and (11) we obtain

$$F \leqslant 4/\lambda^2 \ . \tag{18}$$

We can now employ the quantum speed limit inequality (4) from [18] and merge it with the condition (9) to obtain

$$(\lambda+1)(\Delta X+\Delta X') = |x'-x| \ \geqslant \ \frac{\pi}{2}\max\left\{\frac{\alpha(F)}{\nu(\langle H\rangle - E_0)}, \frac{\beta(F)}{\sqrt{\nu}\Delta H}\right\} \geqslant \frac{\pi}{2}\max\left\{\frac{\alpha(4/\lambda^2)}{\nu(\langle H\rangle - E_0)}, \frac{\beta(4/\lambda^2)}{\sqrt{\nu}\Delta H}\right\}, \tag{19}$$

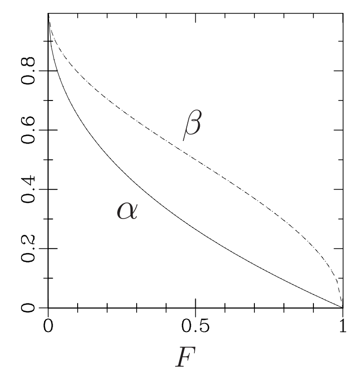

FIG. 2: Plot of the functions $\alpha(F)$ and $\beta(F)$ appearing in Eq. (4).

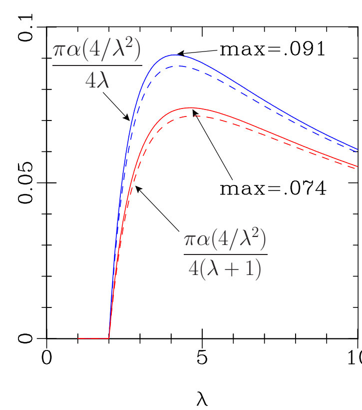

FIG. 3: Plot of the function $\pi\,\alpha(4/\lambda^2)/(4\lambda)$ as a function of λ (blue continuous line). The function α is evaluated numerically according to the prescription of [18]. The same function obtained by approximating α as $\beta^2(F) = 4\arccos^2(\sqrt{F})/\pi^2$ is plotted with a blue dashed line. The red continuous and dashed lines are analogous, but depict the function $\pi\,\alpha(4/\lambda^2)/[4(\lambda+1)]$ that must be employed in the case of biased measurements.

where, as in the main text, we used the fact that α and β are decreasing functions of their arguments, and the fact that the expectation and variances of H over the family ρ_x is independent of x (since H is independent of x). The first term of Eq. (19) together with the first part of the max implies Eq. (3), choosing $\kappa = \sup_\lambda \pi\alpha(4/\lambda^2)/[4(\lambda+1)] \simeq 0.074$, which for unbiased estimation can be replaced by $\kappa = \sup_\lambda \pi\alpha(4/\lambda^2)/[4\lambda] \simeq 0.091$. In the case in which $\Delta X(x) = \Delta X(x') = \Delta X$ we then immediately obtain the bound (2).

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